Riddling, bubbling, and Hopf bifurcation in coupled map systems

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In a simple coupled-map system, we show that the desynchronization of the chaotic attractor occurs through a series of Hopf bifurcations. As a result of the accumulation of these processes, the blowout bifurcation of this coupled map system is a *new* kind of Hopf bifurcation *from a chaotic attractor*. Riddling, bubbling, and characters of this particular manifestation of Hopf bifurcation from a chaotic attractor are studied in this paper. [S1063-651X(99)14811-6]

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Recently, the phenomenon of synchronization of identical chaotic systems coupled in an array has become a very intensely studied subject [1–7]. An important question in this field concerns the form of the basin of attraction of the synchronized state and the bifurcations through which this basin, or the attractor itself, undergoes qualitative changes [2–6,8–10]. Interesting phenomena as riddled basins of attraction [2,4–6,8,10] and on-off intermittency [3,9] have been observed during the destabilization of the synchronous state. Occurrence of the riddled basin of attraction is due to the so-called riddling bifurcation in which an unstable periodic orbit embedded in the synchronous chaotic attractor loses its transverse stability. Different kinds of local bifurcations such as subcritical the pitchfork bifurcation [10], the period-doubling bifurcation and the supercritical pitchfork bifurcation [5] can be candidates of this riddling bifurcation. After the riddling bifurcation, a set of tongues opens from the transversely unstable periodic circle and its dense set of preimages in the synchronous chaotic state [10]. Phase points falling in these tongues will be repelled from the synchronous state on the main diagonal. In case the riddling bifurcation is supercritical, the transversely destabilized orbit will be surrounded by saddle points with unstable manifolds along the invariant subspace [5]. In this case, the orbits wander around in the phase space after they are repelled from the main diagonal. Sooner or later they will be reinjected again into the proximity of the main diagonal. Some of them will be mapped again into these tongues after some steps of iterations, and hence, again leave the neighborhood of the synchronous state. This gives rise to the intermittent burst from the invariant subspace. Since the transverse Lyapunov exponents are all negative, the burst will tend to stop. This intermittent burst is called attractor bubbling [4] and the basin of attraction of this attractor is referred to as local riddling. In this case, the motion of the phase points is restricted by nonlinear mechanism within a region of phase space (the absorbing area) [11] that lies strictly inside the basin of attraction of the synchronous state. With variation of the parameter, the absorbing area of the synchronous chaotic state will grow and/or the basin of attraction of this state will shrink. The boundary of the absorbing area will make contact with the boundary of the basin of attraction at a certain moment. After this boundary crisis [11], the orbits falling in the tongues will eventually escape to the basin of attraction of another attractor or infinity. In case the boundary crisis is before the blowout bifurcation, the synchronous chaotic state attracts a set of points of positive Lebegue measure in the neighborhood of the invariant subspace. However, arbitrarily close to any such point one may find a positive Lebegue measure of points that are repelled by the chaotic attractor. This attractor is referred as a weaker Milnor attractor. And its basin of attraction is globally riddled.

In this paper, a ring of logistic maps with both diffusive and gradient coupling is studied. Such a ring structure is frequently used in the study of biochemical and physiological problems [12]. It is shown that the Hopf bifurcation can be a new candidate of riddling bifurcation in such a system. Unstable periodic orbits embedded in the synchronous chaotic state lose their transverse stabilities through a series of Hopf bifurcations. Due to the accumulation of these partial destabilizations, the blowout bifurcation [9] exhibits to be a Hopf bifurcation *from a chaotic attractor* [7]. This particular manifestation of the Hopf bifurcation has some new character in contrast to the traditional case *from a fixed point or periodic orbit*.

Let us consider the system of three coupled logistic maps in the form:

$$x_{n+1}=f(x_n)+(\epsilon+r)g(y_n,x_n)+(\epsilon-r)g(z_n,x_n), \quad (1)$$

$$y_{n+1}=f(y_n)+(\epsilon+r)g(z_n,y_n)+(\epsilon-r)g(x_n,y_n), \quad (2)$$

$$z_{n+1}=f(z_n)+(\epsilon+r)g(x_n,z_n)+(\epsilon-r)g(y_n,z_n), \quad (3)$$

where the local dynamics is of the form of the logistic map $f(x)=ax(1-x)$ and a represents the controlling parameter of the single map, ϵ and r are the coefficients of diffusive and gradient coupling respectively, $g(x,y)$ is the coupling function satisfying the condition $g(x,x)=0$. With a suitable setting of the coupling coefficients ϵ and r, the three logistic maps can be synchronized. In this case, the system (1) possess an attractor in the symmetry subspace $x_n=y_n=z_n\equiv s_n$.

To determine the transverse stability of the synchronized state, one should let $x_n=s_n+\delta x_n, y_n=s_n+\delta y_n, z_n=s_n+\delta z_n$ and linearize the system (1) about the synchronized state $x_n=y_n=z_n\equiv s_n$. This leads to

$$\Delta_{n+1} = M_n \Delta_n, \tag{4}$$

where $\Delta_n = (\delta x_n, \delta y_n, \delta z_n)$ and the transform matrix M_n is of the form:

$$M_n = \begin{pmatrix} f'(s_n) - 2\epsilon g' & (\epsilon + r)g' & (\epsilon - r)g' \\ (\epsilon - r)g' & f'(s_n) - 2\epsilon g' & (\epsilon + r)g' \\ (\epsilon + r)g' & (\epsilon - r)g' & f'(s_n) - 2\epsilon g' \end{pmatrix} \tag{5}$$

where $f' \equiv df(x)/dx$ and $g' \equiv \partial/\partial x g(x,y)$. This matrix has three eigenvalues. The first one is $\sigma_1 = f'(s_n)$. The corresponding eigenvector is $(1,1,1)$ along the main diagonal. It characterizes the dynamics of the synchronized state along the main diagonal. The two other ones are $\sigma_{2,3} = f'(s_n) - 3\epsilon g' \pm i\sqrt{3} r g'$ which characterize the transverse dynamics of the synchronized state. For an N-periodic synchronous state $\gamma_N = \{x_1, x_2, \ldots, x_N\}$, the criterion for its transverse stability is

$$\prod_{n=1}^{N} |[f'(x_n) - 3\epsilon g']^2 + 3r^2 g'^2| < 1. \tag{6}$$

It should be noted that the two eigenvalues for the transverse dynamics are of complex values. Hence destructions of the transverse stability of the periodic cycles are through Hopf bifurcations. The riddling bifurcation is of a new local dynamics in contrast to the former reported cases [5,6,10]. As a result of the accumulation of this desynchronization of the periodic cycles, the synchronous chaotic state $x_n = y_n = z_n$ becomes transversely unstable when the largest transverse Lyapunov exponent $\Lambda_\perp$ changes its sign. This is called the blowout bifurcation [9]. Here we have two identical transverse Lyapunov exponents for the synchronous chaotic state:

$$\Lambda\perp^{(1,2)} = \lim_{N\to\infty} \frac{1}{N} \ln|\tilde{\sigma}_N^{(1,2)}|, \tag{7}$$

where $\tilde{\sigma}_N^{(1,2)} = \Pi_{n=1}^{N}[f'(x_n) - 3\epsilon g' \pm i\sqrt{3} r g']$ are the two complex eigenvalues of the matrix $\tilde{M}_N = \Pi_{n=1}^{N} M_n$, x_n is a typical orbit which leads to the chaotic attractor. And the blowout bifurcation in this case is corresponding to the crossing of the pair of complex eigenvalues $\tilde{\sigma}_N^{(1,2)}$ through the unit circle. So, we would like to say that here the blowout bifurcation is a Hopf bifurcation from a chaotic attractor. Throughout the rest of this paper we will use a specific form of coupling function $g(x) = x$ to show numerically what happens during the gradual destruction of the synchronous chaotic state.

By determining the transverse stability of the main low-period cycles embedded in the synchronous chaotic attractor [13], a diagram of the stability region in the parameter space (ϵ, r) is plotted in Fig. 1. In region A, all the periodic circles are transversely stable. Hence the synchronous state is absolutely stable in this case. In region B, some of the low period circles are transversely unstable while the transverse

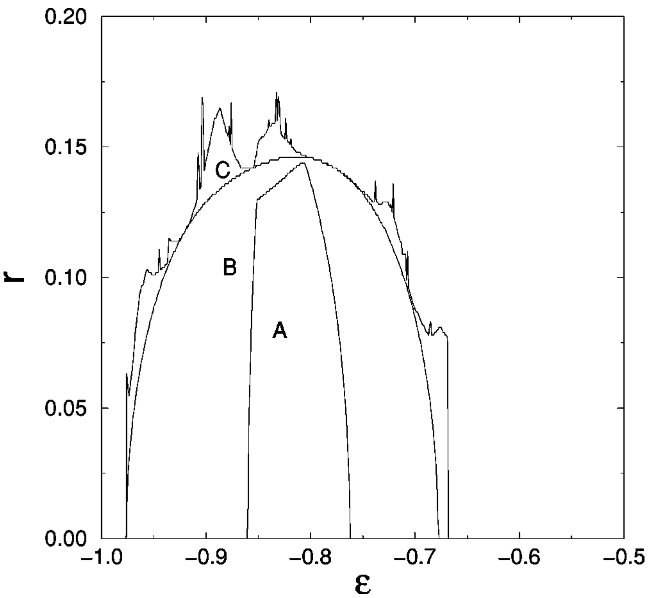

FIG. 1. The stability diagram of the system (1) in the phase plane (ϵ, r) and $a = 3.58$.

Lyapunov exponents are all negative. The synchronous chaotic attractor is stable on the average and attracts a set of points of positive Lebegue measure from its neighborhood. In this case, it may be referred to as a Milnor attractor [14]. The curve dividing regions A and B corresponds to the riddling bifurcation. For the parameter setting used, the boundary crisis [9], where the absorbing area of the synchronous state collides with the basin boundary of this state, occurs after the blowout bifurcation. So, only local riddling (or say attractor bubbling) can be observed in region B. With a small amplitude noise or parameter mismatch, the system exhibits intermittent bursts from the invariant subspace. An example of the basin structure of the synchronous state $x_n = y_n = z_n$ for $\epsilon = -0.84$ and $r = 0.14$ is shown in Fig. 2(a). With further increase of the gradient coupling r, the maximum transverse Lyapunov exponent will change its sign from negative to positive. The curve dividing the regions B and C is corresponding to the zero value of the maximum transverse Lyapunov exponent. In region C, the synchronous state on the main diagonal $x_n = y_n = z_n$ becomes transversely unstable. After the breakdown of the one-dimensional synchronous state, a new three-dimensional attractor appears. The basin of attraction of this new attractor is shown in Fig. 2(b). It can be seen that, immediately after the blowout bifurcation, the basin of attraction remains practically unaffected by the change in the attractor. The motion of a typical orbit on the new attractor is the combination of the four-band chaotic motion along the main diagonal [see Fig. 2(c)] and the circling motion around the main diagonal in the transverse direction [Fig. 2(d)]. The former is the continuation of the chaotic motion on the main diagonal for the synchronous case.

To study the transverse motion of the system (1) just beyond the blowout bifurcation, we first see the evolution of the variable $d_n \equiv \sqrt{(y_n - x_n)^2 + (z_n - y_n)^2 + (x_n - z_n)^2}$ which has the meaning of the instantaneous distance of the phase point (x_n, y_n, z_n) from the main diagonal $x_n = y_n = z_n$. Nu-

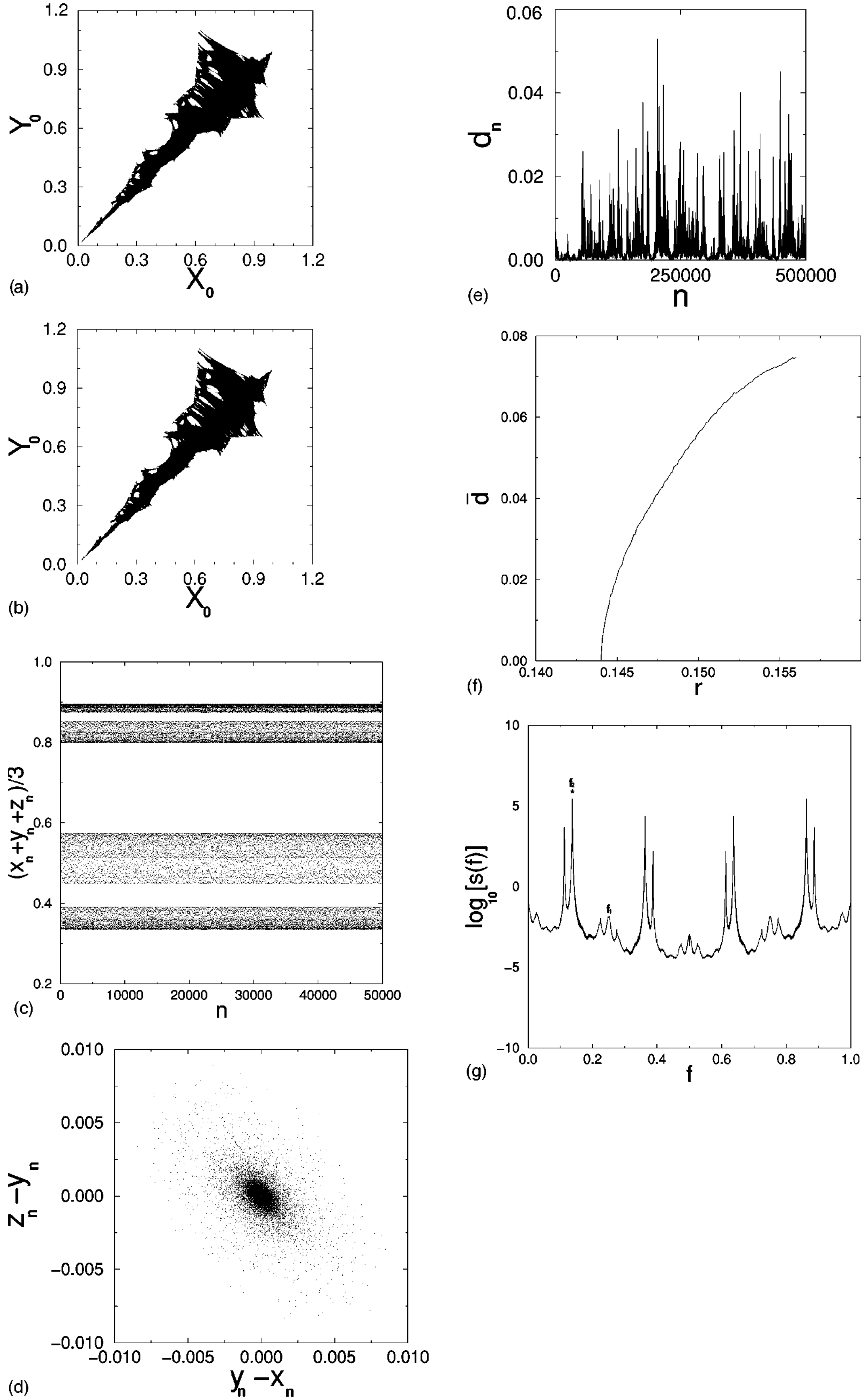

FIG. 2. (a) The basin of attraction of the synchronous state $x_n=y_n=z_n$ in the plane (x_n, y_n). Here the initial value of z_n is set to $(x_n+y_n)/2$; the parameter setting for (a) is $a=3.58$, $\epsilon=-0.84$ and $r=0.14$; (b) attracting basin of the 3-dimensional attractor beyond the blowout bifurcation; (c) $\overline{s_n}\equiv(x_n+y_n+z_n)/3$ vers n; (d) projection of the 3-dimensional attractor on the plane (y_n-x_n, z_n-y_n); (e) temporal evolution of the distance d_n; (f) variation of the mean distance $\overline{d}$ with that of r; (g) the power spectrum of the variable y_n-x_n; the parameter setting for (b)–(g) is $a=3.58$, $\epsilon=-0.84$ and $r=0.1441$. Here the critical value of the parameter r for the blowout bifurcation is $r_c=0.144$.

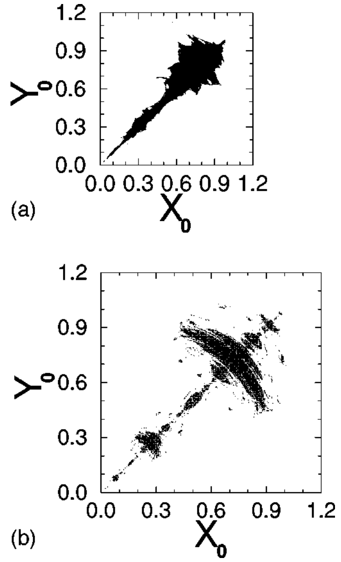

FIG. 3. (a) Local riddled basin for the case of $a=3.65$, $r=0.05$ and $\epsilon=-0.81$; (b) global riddled basin for the case of $\epsilon=-0.65$.

merical results of the variation of d_n with the time n is shown in Fig. 2(e). The long time near zero phase is interrupted irregularly and continuously by the short time large amplitude burst events. This is the typical character of the on-off intermittency [3]. Also, variation of the mean distance $\overline{d} \equiv \lim_{N\to\infty} 1/N \Sigma_{n=1}^{N} d_n$ with that of the parameter setting r is calculated [Fig. 2(f)]. Here we keep the parameter $a=3.58$ and $\epsilon=-0.84$. It can be seen that beyond the blowout bifurcation, the mean distance $\overline{d}$ increases as the power law of the parameter deviation: $\overline{d} \propto (r-r_c)^{\gamma}$ where $\gamma\approx 0.5$ and $r_c \approx 0.144$ for the used parameter setting. This is different from other cases of on-off intermittency [3], where the exponent γ has a different value $\gamma\approx 1.0$. Also, the power spectrum of the variable $y_n - x_n$ is calculated [see Fig. 2(g)]. At least two independent frequencies can be seen from the plot. They are $f_1=0.25$ and $f_2=0.1375$ respectively. Comparing it with that of the variable x_n, we know that the first one $f_1=0.25$ is from the four band chaotic motion of x_n. The other one is newly appearing after the blowout bifurcation. We have known that the occurrence of this blowout bifurcation results from the accumulation of Hopf bifurcations of unstable periodic circles embedded in the chaotic attractor. After the Hopf bifurcation of each periodic circle, a new frequency will appear. So, the new frequency f_2 should correspond to that resulting from the Hopf bifurcation of a certain periodic circle. And, it is also expected that, in general, there should be more than one new frequencies appearing after the blowout bifurcation. For the parameter setting used, the frequencies corresponding to the main low order periodic orbits are: $f_{p_4}=0.1330$, $f_{p_8}=0.1381$ and $f_{p_{16}}=0.1370$. They are almost of the same value. This may be the reason why we observe only one new frequency in the power spectrum of the variable $y_n - x_n$.

Also, for other parameter settings, we can have the case that the boundary crisis, where the boundary of the absorbing area of the synchronous chaotic state collides with the basin boundary of this state, occurs before the blowout bifurcation. There, a bifurcation from the locally riddled basin to the globally riddled basin can be observed. One example of such a bifurcation for the parameter setting $a=3.65$ and $r=0.05$ is shown in Fig. 3. In Fig. 3(a), the basin of attraction of the synchronous state has a fractal boundary. But there are no tongues in this basin of attraction belonging to that of another attractor. Orbits repelled from the main diagonal can never reach the basin boundary. This kind of basin of attraction is referred to as local riddling. In Fig. 3(b), for any point in the basin of attraction of the synchronous chaotic state, in an arbitrarily small region about this point, there is a set of finite measure which belongs to the basin of attraction of another attractor. The basin of attraction of the synchronous state is globally riddled.

In summary, the main results of this paper are as follows. First, comparing to the case reported formerly, the Hopf bifurcation as a new candidate of the local dynamics of the riddling bifurcation can also cause the destruction of the transverse stability of the periodic circles embedded in the synchronous chaotic state. Second, the blowout bifurcation in the studied system is a particular case of a Hopf bifurcation from a chaotic attractor. In contrast to the normal case of a Hopf bifurcation from a fixed point or limited circle, it has two new features: (1) The radius of the circle motion in the transverse direction shows extreme intermittent behavior and its mean value increases as the square root of the parameter deviation; (2) Some new frequencies appear after the blowout bifurcation. They result from the Hopf bifurcation of the periodic circles embedded in the synchronous chaotic state.

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